Notes on Galileo*

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Abstract

In the first section we present a simple argument showing that Galileo's account of his pendulum experiments given in the Discorsi cannot be faithful with regard to his statement of isochronism. In the second section we investigate the sense in which the true, hyperbolic curve of the hanging chain is approximated by a parabolic curve, which Galileo thought would exactly describe ballistic trajectories and hanging chains alike.

1 On Galileo's Exaggerations

That Galileo somewhat exaggerated the outcome of experiments described in his *Discorsi* is often suspected. Leaving alone the question as to why this might happen, it seems useful to also produce some precise quantitative estimations of such suspected exaggerations. This we shall do in this appendix for the famous case concerning the isochronism of the pendulum. Compare e.g. Drake (1990), chapters 1 and 14.[1] Our estimations will be based on the *exact* formula for the period of a pendulum *without friction*.[2]

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[1]For example, with respect to this example S. Drake states on p. 210-211 that "when the arc to the vertical for the pendulum having the wider swing is no more than $25°$, the difference in times for it and the other pendulum is not very great and it keeps on diminishing." After all, the following quantitative estimation shows that such differences are observable after at most 20 swings.

[2]Friction has two effects: 1) It leads to an exponential damping of the amplitudes, 2) it enhances the period by an amount depending on the damping. The first affects our considerations

In a famous part towards the end of the first 'day' of the *Discorsi* (Galilei 1974), Galileo (i.e., Salviati) gives the following account of an experiment:

> "Ultimately, I took two balls, one of lead and one of cork, the former being at least a hundred times as heavy as the the latter, and I attached them to equal thin strings four or five braccia long, tied high above. Removed from the vertical, these were set going at the same moment, and falling along the circumferences of the circles described by the equal strings that were the radii, they passed the vertical and returned by the same path. Repeating their goings and comings a good hundred times by themselves, they sensibly showed that the heavy one kept time with the light one so well that not in a hundred oscillations, nor in a thousand, does it get ahead in time even by a moment, but the two travel with equal pace. The operation by the medium is also perceived; offering some impediment to the motion, it diminishes the oscillations of the cork much more than those of the lead. But is does not make them more frequent, or less so; indeed, when the arcs passed by the cork were not more than five or six degrees, and those of the lead were fifty or sixty, they were passed over in the same times."

Taking for the *braccio* 0.6 meters and hence the length of the pendulum between 2.4-3.0 meters, we see that we talk about periods certainly larger than 3 seconds.

The amplitude α is taken to be the angle between the thread of the pendulum and the vertical (direction of the gravitational field). The exact expression for the period T as function of α is an elliptic integral of first kind whose expansion in

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insofar as we will calculate accumulated phase differences for pendulums of substantially different amplitudes. Hence we must check that the actual damping indeed allows to maintain such a difference in amplitudes for the considered periods of accumulation. Regarding 2) we need to estimate this effect since it threatens to level our calculated phase difference which is solely based on the enhancement of the period with amplitude. Applied, as below, to a situation of two pendulums, one with large amplitude and small damping, the other with smaller amplitude because of stronger damping, we see that *both* pendulums will suffer an enhancement of their periods, albeit from different sources. However, the estimation of the enhancement due to damping is easily done and shows that a levelling of these two effects does not occur. To see this, let σ denote the number of full swings after which the amplitude has dropped by a factor of e^{-1}, the difference ΔT to the undamped period T is then given by $\Delta T/T = (8\pi^2\sigma^2)^{-1}$ (plus higher powers in $(2\pi\sigma)^{-2}$, which we can safely neglect). Hence the corresponding number of swings after which a phase difference of $2\pi/n$ to the undamped pendulum has occurred is given by $\tilde{N}_n = \sigma^2\, 8\pi^2/n$. Note in particular the *quadratic* dependence on σ and the relatively large prefactor $8\pi^2 \approx 79$. This means that even for a considerable damping, like $\sigma = 5$, we would have to wait around 200 full swings to see a phase difference to the undamped pendulum of $2\pi/10$. This is a much smaller effect than the one discussed below.

terms of $\sin(\alpha/2)$ begins as follows (A. Sommerfeld, Mechanics):

$$T(\alpha) = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{1}{4}\sin^2\frac{\alpha}{2} + \frac{9}{64}\sin^4\frac{\alpha}{2} + \cdots\right\}. \tag{1}$$

Hence the period increases with the amplitude resulting in the lead-pendulum falling behind the cork-pendulum. We denote by $N_n(\alpha)$ the smallest integer number of full swings beyond which a pendulum of constant amplitude α will have fallen behind a time of at least T/n against a pendulum of period sufficiently close to $T := 2\pi\sqrt{l/g}$ (i.e. of sufficiently small amplitude, like $3°$). After N_4 full swings the phase difference is at least $\pi/2$ and certainly detectable by be naked eye, since then the pendulums start to move in opposite directions. More careful but still unsophisticated observations should reveal deviations from synchrony by, say, one tenth of T, that is, after N_{10} swings.[3]

By definition of $N_n(\alpha)$ we have

$$N_n(\alpha) = \text{smallest integer} \geq \frac{T}{n\cdot(T(\alpha) - T)}. \tag{2}$$

Using (1) we get for the various values of α and $n = 4$ or $n = 10$:

α	$10°$	$15°$	$20°$	$25°$	$30°$	$35°$	$40°$	$70°$	$80°$
N_4	132	59	33	21	15	11	9	3	2
N_{10}	53	24	14	9	6	5	4	2	1

From these values we infer that a situation with amplitudes $\alpha_{\text{lead}} = 25°$ and $\alpha_{\text{cork}} = 3°$ certainly cannot have appeared synchronous for longer than about 20 full swings.

The situations becomes even more drastic in a later description of a similar experiment, reported shortly after the beginning of the fourth day (Galilei 1974, p 226). In this second experiment two balls of lead are suspended on equally long strings of 4-5 braccia and the periods compared for amplitudes $\alpha_1 = 5°$ and $\alpha_2 = 80°$ (!). Here again the assertion is that no deviations from synchrony could be detected, whereas our values for N_4 show that it must have been clearly apparent after 3 full swings the latest. After 4 full swings the two pendulums will even cross the origin approximately simultaneously with oppositely directed velocities.[4]

[3] For example, by letting two separate experimenters count and voice the passages of zero amplitude for the two pendulums respectively. Such a method is in fact suggested in the *Discorsi* (Galilei 1974, p. 227).

[4] In Galilei (1974), footnote 12 on page 227, S. Drake states that "a disagreement of about one beat in thirty should occur with pendulums of length and amplitudes described here". Unfortunately he did not state how he arrives at this result, which, seen from our analysis, seems to be an underestimation of the real effect by more than a factor of 3.

2 Theory of the Hanging Chain and its Parabolic Approximation

We first describe the modern theory of the hanging chain, called $\mathbf{T}_{\text{ex}}$, in terms of Newtonian concepts, and then its approximation, called $\mathbf{T}_{\text{ap}}$, for small slopes y'. The latter gives rise to parabolic shapes, as the former would if the mass distributions were constant along the horizontal projection of the chain rather than along the chain itself. On the level of physical quantities ("Observables") this approximation corresponds to expansions in terms of $\frac{d}{D}$ to various degrees, depending on the observable, where $2D =$ *horizontal distance of the suspension points* and $d =$ *sag*, i.e. the length of the perpendicular from the horizontal line joining the suspension points to the lower apex.

2.1 The Exact Theory $\mathbf{T}_{\text{ex}}$

We will think of the hanging chain as being given by a function $y(x)$ in a Cartesian xy-plane. A point in this plane is denoted by its coordinates (x, y), so that the curve is the set of points $(x, y(x))$, where x ranges over an interval which we take to be $[-D, D]$. y' and y'' denote the first and second derivatives of y with respect to x.

The fundamental equation for the theory of the chain is obtained from a simple and typical argument based on a local application of the principle of *balance of forces*. To do this, we imagine the chain being cut at $(x, y(x))$ and consider one end. We denote by $F(x)$ the strength of the force that one would have to apply to one end in order to keep the corresponding part of the chain in its place. This is also called the chain's tension. We can decompose $F(x)$ into a horizontal component $F_h(x)$ and a vertical component $F_v(x)$. Since by definition a chain can only support tangential forces, these components must satisfy

$$y'(x) = \frac{F_v(x)}{F_h(x)}. \tag{3}$$

If the external force (gravitation) has no horizontal component, $F_h(x)$ must in fact be independent of x. Otherwise a piece of chain with different strengths of the outward pointing horizontal forces could not stay at rest; hence

$$F_h(x) = F_h = \text{const.} \tag{4}$$

Differentiating (3) once more then leads to

$$y''(x) = \frac{F_v'(x)}{F_h}. \tag{5}$$

It is now easy to express the right hand side of (refeq:B2) as function of x and $y'(x)$, since $F_v(x+dx) - F_v(x)$ must clearly be equal to the weight of the piece of chain between $(x, y(x))$ and $(x+dx, y(x+dx))$. If we denote by μ the mass per unit length of the chain, which we take to be constant[5], then its weight is given by $\mu g\, ds(x)$, where $ds(x) = \sqrt{1+[y'(x)]^2}\, dx$ is the length of the (infinitesimal) piece of chain that we consider. Hence (5) results in

$$\frac{y''}{\sqrt{1+[y']^2}} = \frac{1}{h} := \frac{\mu g}{F_h}\,, \tag{6}$$

which is our fundamental equation defining $\mathbf{T}_{\text{ex}}$ ['ex' for exact].

Upon integration with boundary data $y(x = \pm D) = 0$ one obtains the famous $\cosh$-form[6]

$$y(x) = h\,[\cosh(x/h) - \cosh(D/h)]\,. \tag{8}$$

For an engineer, say, it would be more appropriate to eliminate the non geometric parameter h in favour of the *length* L of the chain, given by

$$L = \int_{-D}^{D} dx \sqrt{1+[y']^2} = 2h\,\sinh(D/h)\,, \tag{9}$$

or its sag

$$d := -y(x=0) = -h\,[1 - \cosh(D/h)] = 2h\,\sinh^2(D/2h)\,, \tag{10}$$

i.e., to solve (9) for $h(L, D)$ or (10) for $h(d, D)$ respectively, and insert this into (8). But this cannot be done in terms of elementary functions. Hence (8)(9) or (8)(10) should be thought of as *implicit* representation of the hanging chain as function of the parameters L, D or d, D respectively.

Finally, the total tension $F(x)$ of the chain is easily computed:

$$F(x) = \sqrt{F_h^2 + F_v^2(x)} = F_h\sqrt{1+[y'(x)]^2} = \mu g h\,\cosh(x/h), \tag{11}$$

which, using (8), can also be read as saying that F grows linearly in y.

[5] The following formula (6) remains valid for variable μ. It then implies that the hanging chain can be made to assume *any* convex shape by letting $\mu > 0$ vary appropriately along the chain.

[6] An equivalent form, obtained by applying the addition laws for $\cosh$-functions, is

$$y(x) = 2h\,\sinh((x+D)/2h)\,\sinh((x-D)/2h)\,. \tag{7}$$

2.2 The Approximating Theory $\mathbf{T}_{\rm ap}$

Galileo's approximative modelling of the hanging chain by a parabola can be understood within the larger context of an *approximation of theories*. It is obtained as *first* approximation of the fundamental equation (6) for small slopes y'. Such approximations clearly make sense only for $y' < 1$, which is just the regime for which the parabolic approximation of the hanging chain is claimed in the relevant part of the *Discorsi* (pages 256-257 of Galilei (1974); see Renn at al. (2001), citations on pages 36-38. Hence we expand the square-root in (6) in terms of powers of y' and truncate the second and all higher powers. But since y' appears already in squared form under the square-root, this amounts to simply replacing this square-root by 1. *From the derivation of (6) it is clear that this is equivalent to taking the linear mass-distribution as homogeneously along the x-axis instead along the proper length. This, in turn, is precisely the [implicit] assumption that underlies the application of Galileo's results on the distribution of moments along a solid and homogeneous cylindrical beam which rests horizontally supported at both ends; see Renn at al. (2001), p. 120.* In first approximation one simply expands (6) in terms of powers of y' discarding the second and all higher powers.

The fundamental equation that defines the approximating theory $\mathbf{T}_{\rm ap}$ is now simply given by:

$$y'' = \frac{1}{h}\,, \tag{12}$$

and for the same boundary data as above one obtains

$$y(x) = \frac{1}{2h}\,\left(x^2 - D^2\right)\,. \tag{13}$$

Formally this corresponds to a quadratic expansion of the $\cosh$-function in (8) in terms of the dimensionful parameter $1/h$, which should be understood as expansion in terms of a dimensionless parameter $(\frac{1}{h})\times$ (intrinsic length) $\cong D/h$. The sag, d, is now given by the simple formula

$$d = \frac{D^2}{2h}\,, \tag{14}$$

which, in contrast to the exact theory, can now be easily solved for h. This allows us to explicitly parameterise the curve by the geometric quantities d and D. An expansion in terms of D/h is hence equivalent to an expansion in terms of d/D.

Note that in general it will not be the case that the exact expressions of an approximating theory are certain approximations of the exact theory, but only that simultaneous expansions in *both* theories coincide up to some order. For example, the expression for the length L in $\mathbf{T}_{\rm ap}$ has the complicated structure

$$L = \int_{-D}^{D} dx\sqrt{1+(x/h)^2} = D\,\left[\sqrt{1+(D/h)^2} + (h/D)\,\mathrm{arsinh}(D/h)\right]\,, \tag{15}$$

but the quadratic expansions of (9) and (15) in terms of d/D (i.e. in terms of D/h and then h eliminated by using (14)) both lead to

$$L = 2D \left[1 + \frac{2}{3} \left(\frac{d}{D} \right)^2 \right] . \tag{16}$$

The same holds for the total tension, which in $\mathbf{T}_{\rm ap}$ takes the form

$$F(x) = \mu g h \sqrt{1 + (x/h)^2}, \tag{17}$$

whereas the quadratic expansions von (11) and (17) in terms of d/D coincide in the following "engineer-formula"

$$F(x) = \mu g \, \frac{D^2}{2d} \left[1 + 2 \left(\frac{x}{D} \right)^2 \left(\frac{d}{D} \right)^2 \right] . \tag{18}$$

Finally we can raise the question how to grade the quality of the approximation of $\mathbf{T}_{\rm ex}$ by $\mathbf{T}_{\rm ap}$. This can be done for each observable (here observables are e.g. $y(x)$, d, $F(x)$ and L) by looking at the orders of the first non-vanishing correction of $\mathbf{T}_{\rm ap}$ by $\mathbf{T}_{\rm ex}$. Let $\mathcal{O}_{\rm ex}$ and $\mathcal{O}_{\rm ap}$ be the values of an observable on "corresponding" [see below] solutions of the fundamental equation of $\mathbf{T}_{\rm ex}$ and $\mathbf{T}_{\rm ap}$ respectively. Then one considers

$$\Delta(\mathcal{O}) := \frac{\mathcal{O}_{\rm ex} - \mathcal{O}_{\rm ap}}{\mathcal{O}_{\rm ex}}, \tag{19}$$

and defines as usual $o(\Delta(\mathcal{O}))$ to be that integer which characterises the leading order in the expansion of $\Delta(\mathcal{O})$ with respect to the expansion parameter (here d/D). The grade $g(\mathcal{O})$ of the expansion can then be defined as

$$g(\mathcal{O}) := o(\Delta(\mathcal{O})) - 1 \, . \tag{20}$$

In our case we obtain

$$g(y(x)) = g(d) = 1, \quad g(F(x)) = g(L) = 3. \tag{21}$$

From the definition of $\mathbf{T}_{\rm ap}$ together with $y' = \sinh(x/h) = x/h + ...$ one could not have expected a grade of approximation better than 1 [linear approximation]. But, as we just saw, the approximation might come out to be much better. This mirrors a well known phenomena in physics: that some formulae "are better than their derivation". In our case this is for example true for the tension (formulae (17)(18)), which deviates from the exact expression only in *fourth order* in d/D (they slightly *underestimate* the real tension).

Finally we wish to comment on the notion of *corresponding solutions*. In order to define a correspondence one has to make a choice of preferred observables whose values uniquely fix solutions of the fundamental equations (6) and (12). Solutions with coinciding values on these observables are then defined to correspond to each other. Such a definition should therefore always be thought of as *relative* to the choice of preferred observables. So, for example, for given horizontal distance $2D$ of the suspension points one may either take the horizontal tension F_h (as we did) or the length L or the sag d to fix the solution. A non-trivial consequence of this general observation is, that the grade of an approximation of some observable will in general *depend* on the choice of preferred observables which are used to fix the solutions.

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